Generic multifractality in exponentials of long memory processes

A. Saichev[1,2] and D. Sornette[3,4,*]

[1]Mathematical Department, Nizhny Novgorod State University, Gagarin Prospekt 23, Nizhny Novgorod, 603950, Russia
[2]Institute of Geophysics and Planetary Physics, University of California, Los Angeles, California 90095, USA
[3]Institute of Geophysics and Planetary Physics and Department of Earth and Space Sciences, University of California, Los Angeles, California 90095, USA
[4]Laboratoire de Physique de la Matière Condensée, CNRS UMR 6622 and Université de Nice-Sophia Antipolis, 06108 Nice Cedex 2, France

(Received 28 February 2006; published 18 July 2006)

We find that multifractal scaling is a robust property of a large class of continuous stochastic processes, constructed as exponentials of long-memory processes. The long memory is characterized by a power law kernel with tail exponent $\varphi+1/2$, where $\varphi>0$. This generalizes previous studies performed only with $\varphi=0$ (with a truncation at an integral scale) by showing that multifractality holds over a remarkably large range of dimensionless scales for $\varphi>0$. The intermittency multifractal coefficient can be tuned continuously as a function of the deviation φ from 1/2 and of another parameter σ^2 embodying information on the short-range amplitude of the memory kernel, the ultraviolet cutoff ("viscous") scale, and the variance of the white-noise innovations. In these processes, both a viscous scale and an integral scale naturally appear, bracketing the "inertial" scaling regime. We exhibit a surprisingly good collapse of the multifractal spectra $\zeta(q)$ on a universal scaling function, which enables us to derive high-order multifractal exponents from the small-order values and also obtain a given multifractal spectrum $\zeta(q)$ by different combinations of φ and σ^2.

DOI: 10.1103/PhysRevE.74.011111 PACS number(s): 64.60.Ak, 02.50.Ey, 89.75.Da

I. INTRODUCTION

Generalizing the cascade models that started with Richardson [1] and Kolmogorov [2], multifractal cascades have been introduced in turbulence [3,4] to model the anomalous scaling exhibited by the moments of the velocity increments in hydrodynamic turbulence (see, for instance [5] and references therein). They have been since applied to many other complex fields including fractal growth processes, geophysical fields, high energy particle physics, astronomy, biology, and finance [6]. The constructions involved in multifractal cascades are based on hierarchical geometries coupled with multiplicative noise and form discrete hierarchical cascades [7]. They have been very useful to highlight a general mechanism for intermittency and multifractality which reflects the presence of intermittent bursts of fluctuations with long-range correlations. Accordingly, the long-range correlations are seen to result from the large-scale structures that impact the smaller scales through a hierarchical cascade. But discrete cascades have limitations and defects such as spurious effects due to the discreteness (scaling holds only for certain scale ratios [8]), nonstationarity, and absence of causality in the time domain (see, however [9]).

Here, we study the multifractal properties of a class of continuous stochastic processes, constructed as exponentials of long-memory processes with power law memory. Previous works briefly reviewed below have been concerned with the case where the power law memory has a tail exponent equal to 1/2, which leads to logarithmically decaying correlation functions and the necessity for a regularization at large time scales, i.e., the introduction of a so-called integral scale. With logarithmic correlation functions, previous works have shown that this class of processes exhibit the property of multifractality. Here, we extend the problem to power law memory with tail exponent $\varphi+1/2$ which can vary arbitrarily above 1/2. As a consequence of the faster-than-logarithmic decay of the correlation function of the process, the property of multifractality cannot hold exactly anymore. We show, however, that multifractality holds over a remarkably large range of dimensionless scales and that the intermittency coefficients can be tuned continuously as a function of the deviation φ from 1/2 of the exponent of the power law memory and of another parameter σ^2 embodying information on the short-range amplitude of the memory kernel, the ultraviolet cutoff scale, and the variance of the white-noise innovations. For this, we present a motivated robust algorithm to determine the exponents $\zeta(q)$ of the multifractal spectrum that we apply on our numerically determined structure functions or moments of the stochastic process. We exhibit a surprisingly good collapse of the multifractal spectra on a universal scaling function, which enables us to derive high-order multifractal exponents from the small-order values. The scaling ansatz is validated by direct numerical evaluations of integral expressions of the moments of orders up to $q=5$. Our results offer an interesting generalization of the class of multifractal random walks introduced recently and provide a physically interpretable source of multifractal intermittency in terms of the parameters φ and σ^2. Our results have potential use in all the fields in which multifractal properties have been discussed in the time domain. For instance, they provide a rational for the approximate multifractal signatures observed in simple agent-based models of social networks [10], without the need to justify an exact logarithmic scaling for the correlation function of the logarithm of the observable.

*Electronic address: sornette@moho.ess.ucla.edu

Our investigation is in part motivated by a recently introduced earthquake model [18,19], deriving from the physics of thermally activated rupture and long memory stress relaxation for earthquakes. In addition to earthquakes, another natural application is finance. Indeed, the initial argument of Ouillon and Sornette [18,19] concerned the time decay of the conditional expectation of the response function, called the Omori law for the decay of the number of aftershocks in the context of seismology. This conditional temporal multifractal response function obtained in [18,19] generalized the prediction for the multifractal random walk discussed in Ref. [15], which was tested empirically on financial time series. Other possible applications include the scientific fields which have been reported to exhibit multifractality, such as hydrodynamic turbulence, meteorological and hydrological time series, and the internet.

The organization of the paper is the following. Section II presents a short review to position the stochastic process which we also define. Section III defines the effective multifractal exponents and presents a simple efficient method for their estimation. Section IV focuses on the scaling properties of the second-order moment and its associated exponent $\zeta(2)$. Section V presents our main results for the higher-order moments and their exponents $\zeta(q)$. Section VI offers some numerical and mathematical insights on the origin of the effective multifractality. Section VII concludes.

II. STOCHASTIC CONTINUOUS PROCESSES AS EXPONENTIALS OF PROCESSES WITH POWER LAW MEMORY WITH ARBITRARY EXPONENTS

Recently, inspired by the logic of the construction of discrete hierarchical cascades, several works have divised genuine stochastic continuous stationary processes which reproduce their main properties [11–17]. In particular, the so-called multifractal random walk (MRW) has been introduced by Bacry, Delour, and Muzy as the only continuous stochastic stationary causal process with exact multifractal properties and Gaussian infinitesimal increments [11]. Sornette *et al.* [15] have shown that the increments $\delta_\tau X(t)$ at finite scale τ of the MRW can be approximated by

$$\delta_\tau X(t) = \int_{t-\tau}^{t} dW_1(t')e^{\omega_\tau(t')}, \tag{1}$$

where $\delta_\tau X(t)=X(t+\tau)-X(t)$, $W_1(t)$ denotes a Wiener process (with unit diffusion coefficient) and $\omega_\tau(t)$ can be expressed as an autoregressive process

$$\omega_\tau(t) = \omega_\tau^0 + \int_{-\infty}^{t} dW_2(t')h_\tau(t-t'), \tag{2}$$

where ω_τ^0 is a constant, $W_2(t')$ denotes a second Wiener process (with unit diffusion coefficient) which is uncorrelated with $W_1(t)$ and the memory kernel $h_\tau(\cdot)$ is a causal function specified by its Fourier transform

$$[\hat{h}_\tau(f)]^2 = 2\lambda^2 f^{-1}\left[\int_0^{Tf}\frac{\sin(t)}{t}dt + O(f\tau\ln(f\tau))\right]. \tag{3}$$

The expression (3) shows that

$$h_\tau(t') \sim K_0\sqrt{\frac{\lambda^2 T}{t'}} \quad \text{for } \tau \ll t' \ll T, \tag{4}$$

where the so-called integral scale T delineates the boundary beyond which the correlation vanishes exactly. This slow inverse square-root power law decay (4) of the memory kernel in Eq. (2) ensures the long-range logarithmic dependence of the correlation function of $\omega_\tau(t)$ [15], which is one important ingredient for the multifractality of $\delta_\tau X(t)$. Schmitt [16] has studied in detail the stochastic process (1) and (2) with a kernel $h_\tau(t')$ exactly given by the square-root power law (4) for t' between scale $\tau=1$ and T, with a smooth regularization for $t<\tau$. Not surprisingly, this process exhibits multifractality in the range of scales between τ and T.

We study the positive stochastic process $\delta_\tau X(t)$ generalizing Eqs. (1) and (2) with Eq. (4), defined by

$$\delta_\tau X(t) = \int_{t-\tau}^{t}\mu(t')dt', \quad \text{with } \mu(t) = \kappa e^{\omega(t)}, \tag{5}$$

with $\omega(t)$ of the form (2)

$$\omega(t) = \int_{-\infty}^{t} dW(t')h(t-t'), \tag{6}$$

where $W(t)$ denotes a Wiener process (with unit diffusion coefficient) and

$$h(t) = \frac{h_0}{(1+x)^{\varphi+1/2}}H(t), \quad x = t/\ell, \tag{7}$$

where ℓ is some "microscopic" characteristic scale, regularizing the singularity of the power law in the propagator $h(t)$ at $t=0$ and $H(t)$ is the unit step Heaviside function ensuring the condition of causality inherent to most applications. Compared with Eq. (1), we do not introduce the noise dW_1 because, if we did, being uncorrelated with dW in Eq. (6), all even-order moments calculated below would receive a constant contribution from it. This would not be different from the result obtained here by removing the stochastic term dW_1 and replacing it by the constant $\kappa>0$. For the sake of simplicity of notations, we thus make the choice of removing dW_1 without loss of generality with respect to the multifractal properties of the process $X(t)$ defined in Eq. (5).

The main departure from the previously cited works is to consider an exponent $\frac{1}{2}+\varphi$ larger than $1/2$ for the power law decay of the memory kernel $h(t)$. As already mentioned, one can prove rigorously [14] that the process (1) and (2) with Eq. (3), whose power law approximation for the memory kernel has the exact exponent $1/2$, i.e., $\varphi=0$, has a logarithmic decaying covariance function of the auxiliary stationary Gaussian stochastic process $\omega(t)$

$$\langle\omega(t)\omega(t+\tau)\rangle \sim \ln\left(\frac{T}{\tau}\right), \quad \tau < T, \tag{8}$$

associated with the multifractal signature of the process $X(t)$ given by

$$\langle[\delta_\tau X(t)]^q\rangle = a(q)\tau^{\zeta(q)}, \quad \text{for } \tau < T. \tag{9}$$

In expression (9), the angle brackets denote a statistical averaging and the multifractal "spectrum" $\zeta(q)$ has the parabolic form

$$\zeta(q) = \left(1 + \frac{\lambda^2}{2}\right)q - \frac{\lambda^2}{2}q^2, \tag{10}$$

where $\lambda^2 = -\zeta''(0)$ is the so-called *intermittency coefficient*. In contrast, the existence of multifractality defined by the nonlinear spectrum $\zeta(q)$ has not been studied previously for the process (5) and (7) with nonzero values $\varphi > 0$. It is not obvious *a priori* that multifractality will be observed because the deviation from 1/2 for the exponent of the power law decay of the memory kernel implies that the covariance function of the stochastic process $\omega(t)$ is no more logarithmic, which was the fundamental reason for the existence of multifractality in the MRW. However, Ouillon and Sornette [18,19] have recently shown that the process (5) and (7) with nonzero values $\varphi > 0$ has robust multiscaling properties. They have derived this process from the physics of thermally activated rupture and long memory stress relaxation for earthquakes, and have shown that this process predicts that seismic decay rates after mainshocks follow approximately the Omori law $\sim 1/t^p$ with exponents $p(M)$ linearly increasing with the magnitude M of the mainshock, in agreement with observations [18,19]. Such multiscaling suggests that the property of multifractality in the sense of Eq. (9) should also be present.

Our investigation is also relevant to other domains, such as the financial markets, which also exhibit similar properties. Actually, the conditional relaxation found by Ouillon and Sornette [18,19], which in part motivated the present work, was first derived for the MRW to describe financial markets [15]. Indeed, in the context of financial time series, the MRW is only one among a rich literature on long-memory processes, from fractionally integrated ARMA and ARCH processes [20–22] to multifractal cascade models [23–29]. The class of fractionally integrated processes does not exhibit multifractality but only a much weaker form of apparent multifractality [30] than reported in our present paper and we do not consider it further. On the other hand, as mentioned briefly in the Introduction, the MRW and our model can be viewed as the stationary continuous-time descendant of the discrete multifractal cascade models [23–29]. They improve on them by removing spurious effects due to discrete scales, nonstationarity, and noncausality.

A significant difference between the process (5) and (7) with nonzero values $\varphi > 0$ and with $\varphi = 0$ is that no integral scale T is needed to regularize the theory. Furthermore, all moments $\langle[\delta_\tau X(t)]^q\rangle$ are finite for $\varphi > 0$, whereas all the moments of order $q > q_*$ for $\varphi = 0$, where q_* satisfies to equation $\zeta(q_*) = 1$, are mathematically infinite. This divergence signals that the probability density function (PDF) of the increments $\delta_\tau X(t)$ is heavy-tailed with an exponent equal to or smaller than q_*. Using extreme value considerations to tackle the competition between the distribution of the noise in the ω process and the long-range correlation, Muzy *et al.* showed that the observable tail exponent μ is smaller than the value q_* given by $\zeta(q_*) = 1$ by a very large amount and is actually determined by $\mu = q^\dagger$ where $q^\dagger$ is the solution of $D[\alpha(q^\dagger)] = 0$ [31]. The absence of divergence of all moments of the PDF for $\varphi > 0$ excludes a heavy tail but not fat tails of the PDF of increments, such as stretched exponentials (which are known to approach arbitrary well any power law, see [32] and Chap. 2 of [33]). In other words, the existence of all the moments does not exclude the possibility of a fat tail in the form of a stretched exponential PDF. Note that our model has by definition an intrinsic ultraviolet scale ℓ. It is then worthwhile to mention that all moments of MRW processes observed at noninfinitesimal resolution (in other words, observed at a finite resolution) also exist and are finite. It is only when taking the continuous limit that all the moments of MRW processes diverges above the order q^* discussed above. The detailed exploration of the exact shape of the PDF of our model and of the MRW at finite resolutions is a critical question in turbulence and financial applications and deserves a separate treatment which will be presented elsewhere. For our present purpose, it is sufficient to state that, if multifractality exists for $\varphi > 0$, it may be observed for any $q \geqslant 0$.

In addition to φ, the other key parameter controlling the multifractal properties of the process (5) and (7) will be shown to be

$$\sigma^2 = \int_0^\infty h^2(t)dt = h_0^2 \frac{\ell}{2\varphi}. \tag{11}$$

Note the divergence of σ^2 for $\varphi \to 0$, for which the integral scale must be reintroduced to regularize the theory.

III. DEFINITION AND DETERMINATION OF EFFECTIVE MULTIFRACTAL EXPONENTS

A. Definitions and notations

A general theoretical characterization of the increments (5) is offered by the moment functions defined by

$$M(t_1, \ldots, t_q) = \left\langle \prod_{r=1}^{q} \mu(t_r) \right\rangle = \langle\mu^q\rangle \prod_{\substack{i=1 \\ k=i+1}}^{q} G(x_k - x_i) \tag{12}$$

of the lognormal density $\mu(t) = \kappa e^{\omega(t)}$ defined in Eq. (5). In Eq. (12), we use the following notations: $x_i = t_i/\ell$,

$$\langle\mu^q\rangle = \kappa^q e^{\sigma^2 q^2/2}, \quad G(y) = e^{-\sigma^2 d(y)}, \quad d(y) = 1 - C(y), \tag{13}$$

where $C(t)$ is the normalized $[C(0) = 1]$ covariance function of the Gaussian process $\omega(t)$ defined in Eq. (6),

$$C\left(\frac{\tau}{\ell}\right) = \frac{1}{\sigma^2}\int_0^\infty h(t)h(t+\tau)dt. \tag{14}$$

We start our investigation of the multifractal properties of the increments $\delta_\tau X(t)$ defined in Eq. (5) by calculating the moment $\langle[\delta_\tau X(t)]^q\rangle$ for $q = 2$ and checking if the power law scaling of the form (9) holds with an exponent $\zeta(2)$ smaller than 2. If this is the case, and from the fact that $\zeta(0)=0$ and $\zeta(1)=1$, we can conclude that $\zeta(q)$ is a nonlinear function of q, the hallmark of multifractality. The fact that $\zeta(1)=1$ results from the positivity of the stationary density $\mu(t)$.

In order to get the exponent $\zeta(2)$, let us study the normalized second moment

$$S_2(y) = \frac{1}{\ell^2\langle\mu^2\rangle}\langle[\delta_\tau X(t)]^2\rangle = \int_0^y dx_1 \int_0^y dx_2 G(x_2 - x_1). \tag{15}$$

For the numerical analysis of Eq. (15), we use the more convenient representation

$$S_2(y) = 2\int_0^y (y-x)G(x)dx. \tag{16}$$

The question we address is whether and how $S_2(y)$ can be approximated by the power law

$$S_2(y) = A_2 y^{\zeta(2)}. \tag{17}$$

B. Properties associated with the second-order moment

First, let us determine the conditions on $G(x)$ for which the form (17) is exact. For arbitrary behaviors of $S_2(y)$, one can always introduce a local exponent defined by

$$\zeta(2,y) = \frac{d \ln S_2(y)}{d \ln y}, \tag{18}$$

which recovers $\zeta(2,y)=\zeta(2)$ if the power law (17) holds exactly. In the general case where $S_2(y)$ is not a pure power law, we can write

$$\zeta(2,y) = \frac{2}{2-\Delta(y)}, \tag{19}$$

where

$$\Delta(y) = \frac{2X(y)}{y}, \quad X(y) = \frac{\int_0^y xG(x)dx}{\int_0^y G(x)dx}. \tag{20}$$

Thus $\zeta(2,y)$ is independent of y and the scaling law (17) is exact if $X(y)$ is proportional to y, where $X(y)$ has the interesting interpretation of being the barycenter of the segment $[0,y]$ whose mass density is $G(y)$.

It is easy to show that $X(y)\sim y$ if and only if $G(y)$ is a constant or a pure power law $G(y)\sim y^{-\lambda^2}$, with some exponent that we denote $0<\lambda^2<1$ for a reason that will be clear soon. In the former case, $X(y)=y/2$, $\Delta(y)=1$, which yields the nonfractal scaling $\zeta(2,y)=2$. From Eq. (13), we see that $G(y)$ is a constant if the covariance function $C(\tau)$ of the Gaussian process $\omega(t)$ defined in Eq. (14) is also a constant that is $dC(\tau)/d\tau=0$. Taking the derivative of Eq. (14) with respect to τ, integrating by part, and equating to zero for arbitrary values of τ imposes the condition $h(0)=0$, which just means that the measure $\mu(t)$ is uniform, hence the trivial exponent $\zeta(2,y)=2$. The other case $G(y)\sim y^{-\lambda^2}$ is more interesting. This yields

$$X(y) = \frac{1-\lambda^2}{2-\lambda^2}y \to \Delta(y) = 2\frac{1-\lambda^2}{2-\lambda^2} \to \zeta(2,y) = 2-\lambda^2. \tag{21}$$

From Eq. (13), we see that this case corresponds to $C(\tau)$ being an exact logarithmic function of τ, which is the property already mentioned above with Eq. (8) at the origin of the exact multifractality of the process $\delta_\tau X(t)$.

This discussion shows that, for $\varphi>0$, the scaling given by Eq. (17) of the second-order moment can only hold approximately at best. In particular, the local exponent $\zeta(2,y)$ has a simple regular behavior at the two boundaries $y\to 0$ and $y\to\infty$. Indeed, due to the limits

$$\lim_{x\to 0} G(x) = 1, \quad \lim_{x\to\infty} G(x) = \langle\mu\rangle^2/\langle\mu^2\rangle, \tag{22}$$

we have

$$S_2(y) \simeq y^2 \ (y\to 0), \quad S_2(y) \simeq \frac{\langle\mu\rangle^2}{\langle\mu^2\rangle}y^2 \ (y\to\infty), \tag{23}$$

so that

$$\lim_{y\to 0}\zeta(2,y) = \lim_{y\to\infty}\zeta(2,y) = 2. \tag{24}$$

Borrowing the terminology from hydrodynamic turbulence, these two limits (24) imply the existence of an effective "viscous" scale τ_v and of an integral scale τ_i, such that if $\tau \lesssim \tau_v$ and $\tau \gtrsim \tau_i$, then the increments (5) are not multifractal. Note that the limit $y\to\infty$ is attained by taking $\ell\to 0$ at fixed τ since $y=\tau/\ell$. The limit $\ell\to 0$ thus recovers a trivial regime, in contrast with the MRW in which the role of ℓ is played by the scale of resolution which, when going to zero yields a nontrivial genuine multifractal limit.

In the sequel, we will be interested in the "inertial range" which exists if $\zeta(2,y)$ is a very slow function of the dimensionless variable $y=\tau/\ell$ over the interval from some $y_0>1$ to some $y_1 \gg y_0 > 1$. In this case and if $\zeta(2,y)<2$, we will be entitled to speak about the multifractal behavior of the random process $\delta X_\tau(t)$ for some wide range of scales and define the effective exponent $\zeta(2)$ as the minimal value

$$\zeta(2) = \min_y \zeta(2,y), \tag{25}$$

that we derive below as an optimal and efficient definition. Having obtained $\zeta(2)$, we can obtain a first estimation of the intermittency coefficient λ^2 through the relation

$$\lambda^2 = 2 - \zeta(2), \tag{26}$$

which assumes that the parabolic dependence (10) holds.

C. Method of determination of the effective exponent $\zeta(2)$

As we pointed out, the absence of exact multifractility for $\varphi>0$ does not prevent the process from exhibiting approximate multifractal scaling which, for all practical purposes, can be undistinguishable from an exact one. Indeed, empirical data is always noisy and power law scaling is always sampled on a finite (often small) range of scales [34–36]. The experimentally relevant question is thus to define scaling from an operational viewpoint consistent with what is done empirically. We now describe a natural and robust determination of the approximate scaling which will be seen to link intrinsically the definition and determination of the exponent $\zeta(2)$ with the existence of an integral time scale L defined from the range over which the approximate power law scaling holds.

For this, let us consider a y-interval $[y_1, y_2]$. We define $\zeta(2, y_1, y_2)$ over this interval as

$$\zeta(2,y_1,y_2) = \arg\min_{\zeta} \int_{y_1}^{y_2} dy \| \ln S_2(y) - A - \zeta \ln(y) \|^2. \tag{27}$$

In other words, expression (27) determines $\zeta(2, y_1, y_2)$ as the exponent ζ which minimizes in a OLS sense the distance between $\ln S_2(y)$ and a straight line in the $\ln y$ variable over the interval $[y_1, y_2]$. Let us now introduce the parameter η, which measures the precision with which an approximate power law scaling is qualified. Specifically, a power law with exponent $\zeta(2, y_1, y_2)$ is qualified if

$$\min_{\zeta} \int_{y_1}^{y_2} dy \| \ln S_2(y) - A - \zeta \ln(y) \|^2 < \eta. \tag{28}$$

Otherwise, the power law scaling is rejected. For a fixed η, we scan all possible values of y_1 and y_2 for which condition (28) is verified and we select the couple (y_1, y_2) such that the range y_2/y_1 is maximum. We thus obtain an approximate power law scaling with apparent exponent $\zeta(2, y_1, y_2)$ within the confidence or noise level η over the maximum range $[y_1, y_2]$. We believe that this procedure embodies in a precise way the general fitting procedure of experimental data.

Let us study the limit $\eta \to 0$. In the case where $S_2(y)$ is not a pure power law [and thus $\ln S_2(y)$ is not a perfect straight line in $\ln y$], the condition (28) imposes that $y_2/y_1 \to 1$. Therefore, $\zeta(2, y_1, y_2 \to y_1)$ determined by Eq. (27) yields the local slope of the function $\ln S_2(y)$ in the variable $\ln y$, in other words

$$\zeta(2,y_1,y_2 \to y_1) = \left. \frac{d \ln S_2}{d \ln(y)} \right|_{y_1}. \tag{29}$$

Consider now the function

$$f_{y_1,y_2\to y_1}(y_1) = \ln S_2(y_1) - \zeta(2,y_1,y_2 \to y_1)\ln(y_1). \tag{30}$$

This function $f_{y1,y2\to y1}(y_1)$ has at least one minimum in the interval $y \in [\ell, +\infty]$, which we call y_0. Close to its minimum, the function $f_{y1,y2\to y1}(y_1)$ can be expanded up to second order to obtain

$$f_{y_1,y_2\to y_1}(y_1) = A + (1/2)a[\ln(y_1) - \ln(y_0)]^2, \tag{31}$$

where

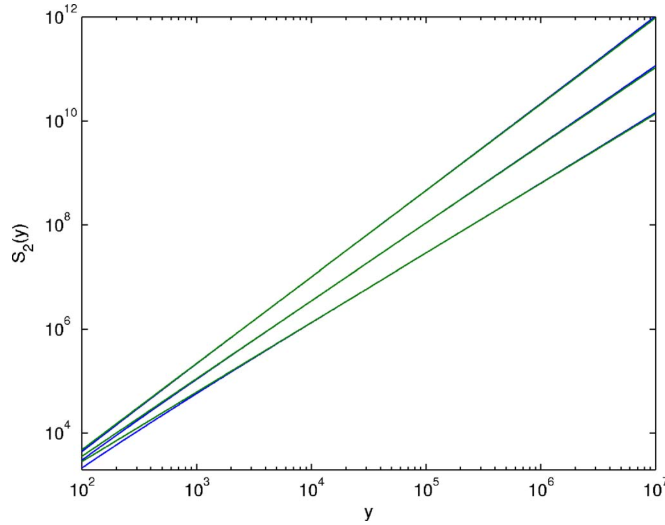

FIG. 1. (Color online) Log-log plot of the second-order moment $S_2(y)$ (16) and its power approximation (17) for φ=0.01 and σ^2=20;30;40 (top to bottom). The corresponding exponents are equal to $\zeta(2)$=1.66;1.49;1.34.

$$a = d^2 \ln S_2/d^2 \ln(y)|_{y_0} \tag{32}$$

is the second-order derivative of $\ln S_2$ with respect to $\ln y$, estimated at y_0. It is clear that, for small finite values of η, the largest range y_2/y_1 is obtained for $y_1 = y_0$ (we assume here that f is convex so that the local minimum is the global minimum; for the range of y's in the examples we have investigated, we have found the convexity condition to always hold). In addition, $f_{y1,y2\to y1}(y_1)$ does not change appreciably over a range of $\ln(y_1)$ proportional to $1/a^{1/2}$. Thus the smaller a is, the larger is the range over which $f_{y1,y2\to y1}(y_1)$ will be almost constant and thus over which the exponent $\zeta(2)$ will be well-defined and constant. This reasoning provides the algorithm to measure the approximate exponent $\zeta(2)$ for a given data set, which we are going to use in a systematic way. It is given by Eq. (29), where y_1 is chosen equal to the argument y_0 such that the second derivative (32) is zero (or reaches the minimum positive value over the whole range available when zero is not crossed). This is equivalent to searching for the exponent (29) which takes the smallest possible positive value over the range of study according to Eq. (25).

IV. SCALING OF THE SECOND-ORDER MOMENT S_2

We study the process (5), whose properties are controlled by the two key parameters φ defined in Eq. (7) and σ^2 defined in Eq. (11). We calculate the second-order structure function $S_2(y)$ defined in Eq. (16) with $y = \tau/\ell$, where $G(x)$ is obtained from expressions (13) and (14).

As a first example, we fix φ=0.01 and scan σ^2=20;30;40. The corresponding structure functions $S_2(y)$ are plotted in Fig. 1 as a function of $y = \tau/\ell$ in log-log scales, together with the best power law fits shown as straight lines. A superficial examination suggests an excellent scaling behavior over at least four orders of magnitude, with an expo-

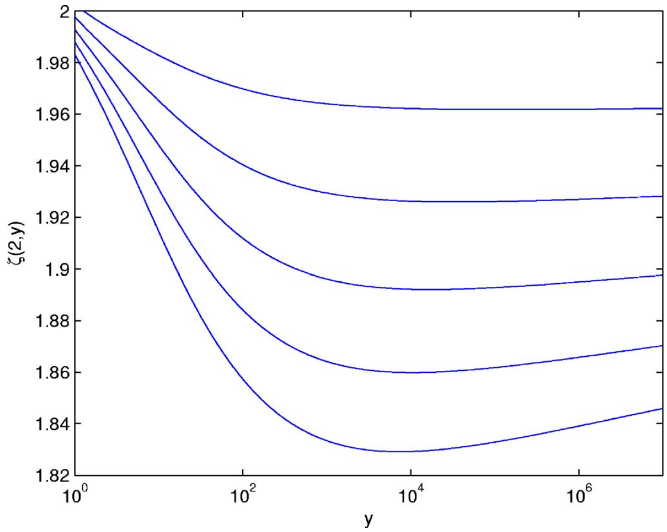

FIG. 2. (Color online) Local exponent $\zeta(2,y)$ as a function of $y=\tau/\ell$ for $\sigma^2=10$ and $\varphi=0.002;0.004;0.006;0.008;0.01$ (top to bottom).

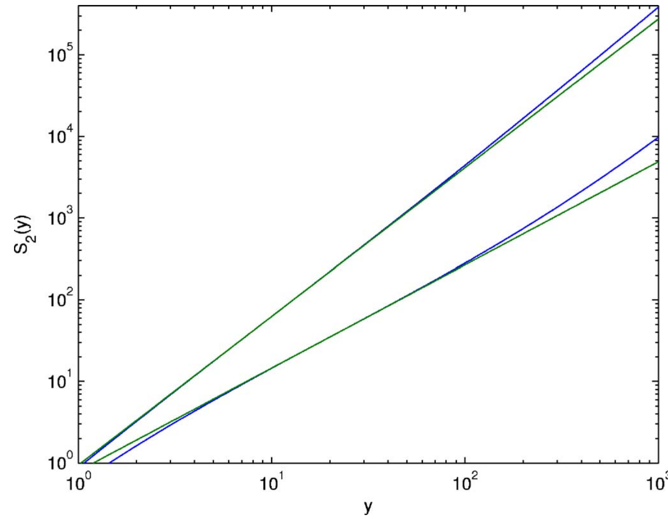

FIG. 3. (Color online) Log-log plot of second-order moment $S_2(y)$ calculated using Eq. (16) and its power approximation (17) for $\varphi=0.5$ and $\sigma^2=1;5$ (top to bottom). The corresponding effective exponents are, respectively, $\zeta(2)=1.82;1.26$.

nent which clearly varies with σ^2 from 1.66 to 1.34 when σ^2 goes from 20 to 40. This property adds to the MRW and previous multifractal process with $\varphi=0$ and, as we are going to explore in some length, allows us to control the multifractal properties continuously as a function of σ^2 as well as φ. Figure 2 examines more precisely the nature of the apparent power law behavior depicted in Fig. 1 by plotting the local exponent $\zeta(2,y)$ defined in Eq. (18) as a function of y in log-scale in the range $y\in[1,10^7]$, for a fixed $\sigma^2=10$ and varying values of $\varphi=0.002$ to 0.01. The first important message of this figure is that the exponent $\zeta(2,y)$ is approximately constant over a large range of y values, all the more so, the closer φ is to zero (this later property is of course not a surprise since $\varphi\to 0$ recovers previously known multifractal processes). Interestingly, this approximately constant value for $\zeta(2,y)$ has a rather large dependence on φ itself, showing again that we can control the intermittency parameter by changing φ as well as σ^2. Another important observation is the sharp variation of $\zeta(2,y)$ on the left-hand side of the range, suggesting a rather well-defined "viscous scale" τ_v, which we characterize as the boundary between the region of approximate constancy of $\zeta(2,y)$ around the definition (25) and the region of sharp increase of $\zeta(2,y)$ as y decreases below the minimum (25). In the examples shown in Fig. 1, τ_v is in the range $10^2-10^3\ell$. The closer φ is to 0, the smaller is the viscous scale τ_v and the better is the multifractal scaling. In contrast, it is not obvious to identify the integral scale over the interval shown in Fig. 1 as the increase of $\zeta(2,y)$ towards 2 has not yet occurred appreciably even up to $y=10^7$, ensuring a rather nice approximate scaling with anomalous multifractal exponent $\zeta(2)<2$. Another way to express this observation is that the "inertial regime" over which the scaling of the second-order moment holds has no well defined boundaries. Therefore the transition to the integral scale is smooth, a property also documented in hydrodynamic turbulence [5].

Figure 3 is the same as Fig. 1 but for a large value of $\varphi=0.5$ and with $\sigma^2=1$ and 5. It shows that the scaling law for S_2 still holds over approximately 2 decades in the horizontal y scales (and of course more in the vertical scale). A plot analogous to Fig. 2 for these values of φ and σ^2 shows obviously larger deviations from a constant behavior (not shown). As we discussed in the previous section, the important message is that we have a rather large latitude in changing φ and σ^2 to exhibit a reasonable (with respect to standard experimental precision) multifractal scaling.

Figure 4 shows the other limit of a small value of $\varphi=0.001$ for a large range of values of $\sigma^2=100-500$. While we expect indeed that the multifractal scaling should extend on large ranges as φ decreases to zero, the most remarkable fact is that the exponent $\zeta(2)$ can be continuously adjusted at will from 2 all the way to close to 1 by varying σ^2 at fixed φ. Note also the large range of y scale over which the apparent

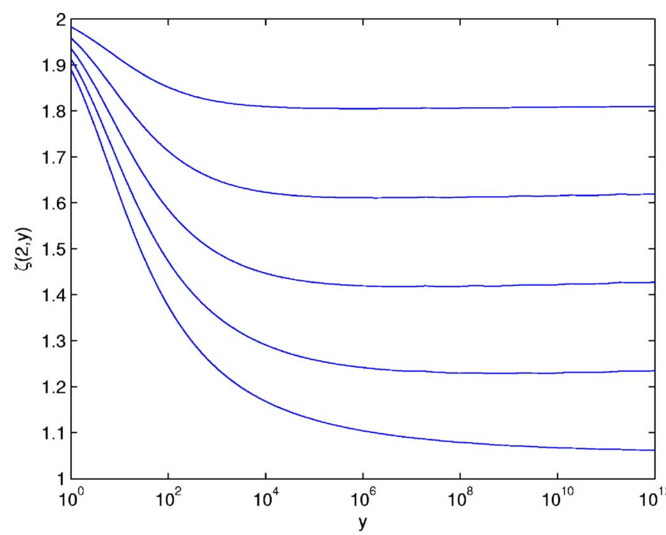

FIG. 4. (Color online) Dependence of the local exponent $\zeta(2,y)$ as a function of $y=\tau/\ell$ in log-scale, for $\varphi=0.001$ and $\sigma^2=100;200;300;400;500$ (top to bottom).

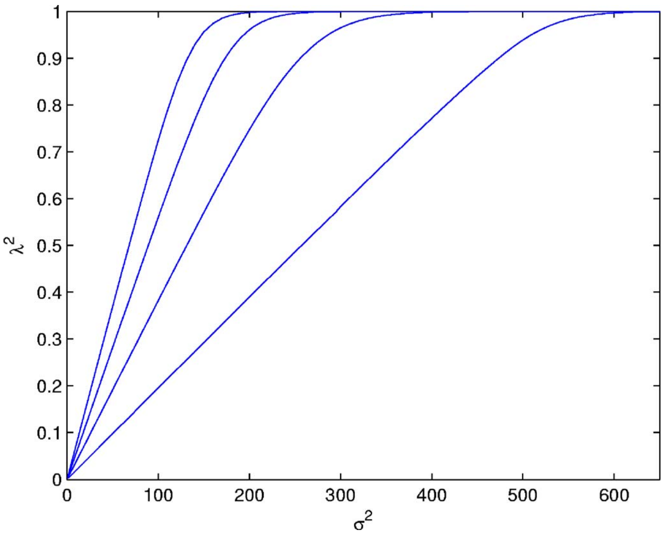

FIG. 5. (Color online) Dependance of the intermittency coefficient $\lambda^2=2-\zeta(2)$ as a function of σ^2 for different values of $\varphi=0.01-0.04$ (bottom to top).

exponent $\zeta(2,y)$ remains approximately constant: for instance, for $\sigma^2=100, \varphi=0.001$, we measure a well-defined constant exponent $\zeta(2)=1.81$ over eight orders of magnitude. Even for these very large ranges of scales, the known existence of an integral scale and the transition to the normal value $\zeta(2)=2$ is not seen.

Figure 5 gives a synopsis of the dependence of the effective exponent $\zeta(2)$ as a function of the two control parameters φ and σ^2. Actually, we show instead the "intermittent coefficient" $\lambda^2=2-\zeta(2)$ first introduced in Eq. (26), in analogy with the parabolic multifractal spectrum (10). The most important observation is that λ^2 increases linearly with σ^2 before saturating to 1 asymptotically for large σ^2's. The upperbound 1 for λ^2 results from the following property of the second-order moment S_2 obtained from the definition (16):

$$\frac{d^2S_2(y)}{dy^2} = 2G(y) > 0. \tag{33}$$

The fact that the second-order derivative of $S_2(y)$ is positive means that $S_2(y)$ increases faster than a linear function of y, hence

$$\zeta(2) > 1 \rightarrow \lambda^2 < 1. \tag{34}$$

V. HIGHER-ORDER MOMENTS, UNIVERSAL SCALING FUNCTION, AND MULTIFRACTAL SPECTRA

A. Definition and determination of the higher-order moments

We have also investigated the higher-order moments S_3, S_4, and S_5 up to order 5 of the increments $\delta_\tau X(t)$ defined in Eq. (5). Moments of arbitrary orders can be obtained from formulas generalizing expression (16) for S_2, as follows:

$$S_q(y) = q(q-1)\int_0^y (y-x)G_q(x)dx, \tag{35}$$

where, for $q>2$,

$$G_q(x) = G(x)\int_0^x du_1 \cdots \int_0^x du_{q-2} \times \prod_{\substack{i=1\\ j=i+1}}^{q-2} G(x_i)G(u-x_i)G(x_i-x_j). \tag{36}$$

The corresponding local exponents for the higher-order moments are defined analogously to Eq. (18) and (19) as

$$\zeta(q,y) = \frac{2}{2-\Delta_q(y)}, \tag{37}$$

where

$$\Delta_q(y) = \frac{2X_q(y)}{y}, \quad X_q(y) = \frac{\int_0^y xG_q(x)dx}{\int_0^y G_q(x)dx}. \tag{38}$$

Using these relations, we adopt the definitions generalizing Eq. (25) for the effective exponents

$$\zeta(q) = \min_y \zeta(q,y) \tag{39}$$

associated with the effective power laws

$$S_q(y) = A_q y^{\zeta(q)}, \quad A_q = S_q(y_m)y_m^{-\zeta(q)}, \tag{40}$$

where y_m is the value of y which makes $\zeta(q,y)$ minimum.

B. Universal scaling ansatz

Our numerical calculations of the higher-order moments ($q \leqslant 5$) show that excellent scaling is observed for these moments over a wide range of the dimensionless variable y, similarly to the case of the second-order moment S_2 presented in Figs. 1–3 [37]. This allows us to determine the dependence of the effective multifractal spectrum $\zeta(q)$ defined by Eq. (39) with respect to q, as well as σ^2 and φ. For this, it is convenient to use the parabolic spectrum (10) as a proxy to extract an effective (*a priori*) q-dependent intermittent coefficient defined by

$$\lambda^2(q;\sigma^2,\varphi) = 2\frac{q-\zeta(q)}{q(q-1)}. \tag{41}$$

We also make explicit in the notation $\lambda^2(q;\sigma^2,\varphi)$ the dependence on the two parameters σ^2 and φ. Similarly to the case of the second-order moment, expression (35) allows us to show that $\zeta(q)$ and $\lambda^2(q;\sigma^2,\varphi)$ satisfy the following inequalities:

$$\zeta(q) > 1 \rightarrow \lambda^2(q;\sigma^2,\varphi) < 2/q, \tag{42}$$

which generalizes Eq. (34) for arbitrary q's.

Our detailed numerical calculations suggest the conjecture that the dependence of $\lambda^2(q;\sigma^2,\varphi)$ with respect to σ^2,φ can be factorized as follows: $\lambda^2(q;\sigma^2,\varphi)=2\Lambda(aq)/q$, where the factor $a=a(\sigma^2,\varphi)$ depends only on the parameters σ^2 and φ and not on q, and the function $\Lambda(x)$ is monotonously increas-

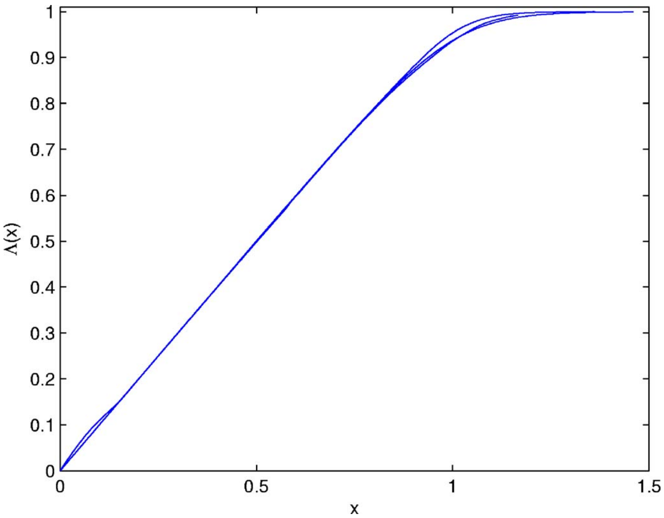

FIG. 6. (Color online) Plot of the universal scaling function $\Lambda(x)$, obtained from relations (41) and (43) and the numerical calculation of the dependence of the effective multifractal exponents $\zeta(q)$ as a function of σ^2, for $\varphi=0.001$ and $q=2;3;4$. The slight discrepancies between the curves in the neighborhood of $x=1$ can be attributed to some systematic errors of numerical calculations.

ing. Moreover, the numerical calculations of the intermittent coefficients show that it is an excellent approximation to represent $a(\sigma^2,\varphi)$ as a linear function of σ^2, i.e., $a=b(\varphi)\sigma^2$. This provides the following universal scaling law for the generalized intermittency coefficients

$$\lambda^2(q;\sigma^2,\varphi)=\frac{2}{q}\Lambda(b\sigma^2 q). \tag{43}$$

The factor b, which is independent of q and of σ^2 according to the scaling ansatz (43), can be determined from the intermittency coefficient $\lambda^2(q=2;\sigma^2,\varphi)=\Lambda(2b\sigma^2)$ for $q=2$, previously reported. Since $\lambda^2(q=2;\sigma^2,\varphi)$ is a linear function of σ^2 for not too large σ^2's as shown in Fig. 5, this implies that the function $\Lambda(x)\sim x$ is linear for small x's. Defining b such that, for $x\to 0$, we have $\Lambda(x)\simeq x$, we obtain

$$b(\varphi)\simeq\frac{\lambda^2(q=2;\sigma^2,\varphi)}{2\sigma^2}\quad(\lambda^2\ll 1). \tag{44}$$

Taking, for instance, $\sigma^2=20$ for which all intermittency coefficients remain small, we find that $b(\varphi)$ is very well represented by the following power law dependence:

$$b\simeq\alpha\varphi^{\beta},\quad\alpha=0.58,\quad\beta=0.92. \tag{45}$$

Figure 6 plots the reconstructed functions $\Lambda(x)=(q/2)\lambda^2(q;\sigma^2,\varphi)$ for different orders q. The excellent collapse provides a first validation of the scaling ansatz (43).

C. Multifractal spectra

The general scaling ansatz (43) allows us to express the multifractal spectrum $\zeta(q)$ under the following form:

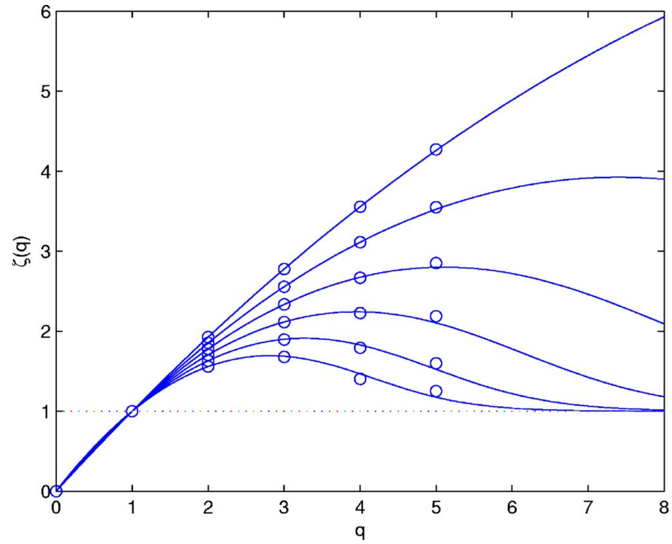

FIG. 7. (Color online) Universal multifractal spectra $\zeta(q)$ for $\varphi=0.004$ and $\sigma^2=10;20;30;40;50;60$ (top to bottom). We compare two different methods for estimating $\zeta(q)$: (i) the circles are the direct numerical integration of Eqs. (35) and (36); and (ii) the continuous lines are obtained by using Eq. (46) with the scaling function $\Lambda(x)$ constructed as in Fig. 6 for $q=2$.

$$\zeta(q)=q+(1-q)\Lambda(b\sigma^2 q). \tag{46}$$

Moreover if, for some particular multifractal phenomenon, the intermittency coefficient $\lambda^2=\lambda^2(2)$ is small compared to 1, then one can rewrite expression (46) in the more universal form

$$\zeta(q)=q+(1-q)\Lambda(\lambda^2 q/2), \tag{47}$$

whose dependence on σ^2 and φ is completely embedded in that of λ^2. This implies that the knowledge of the intermittency coefficient λ^2 together with the scaling function $\Lambda(x)$ allows one to determine the multifractal spectrum for arbitrary orders even for $q\gg 1$.

Our prediction (47) can be checked by direct numerical calculations of the moments up to order 5 that we have performed. Figure 7 plots the multifractal spectra $\zeta(q)$ for $q=0$ to 5 obtained by two methods: (i) the circles are the direct numerical integration of Eqs. (35) and (36) of the stochastic process for $\varphi=0.004$ and different values of σ^2; and (ii) the continuous lines are obtained by using Eq. (46) with the scaling function $\Lambda(x)$ constructed as in Fig. 6 for $q=2$ and $\varphi=0.001$ and applied to the case $\varphi=0.004$. The agreement is good, even for the large values $q=5$. This illustrates that different combinations of φ and σ^2 with fixed $b\sigma^2$ give the same value of the intermittency coefficient λ^2 (for λ^2 not too large) according to Eq. (43) and thus the same multifractal spectrum $\zeta(q)$.

Note that $\zeta(q)$ becomes nonconcave for large σ^2's at large q's, a property which is excluded for exact multifractal scaling by the Hölder inequality applied to the moments $\langle[\delta_\tau X(t)]^q\rangle$. The absence of concavity in a certain range of parameters reflect the fact that multifractality is not an asymptotic property observed at small or large values of the

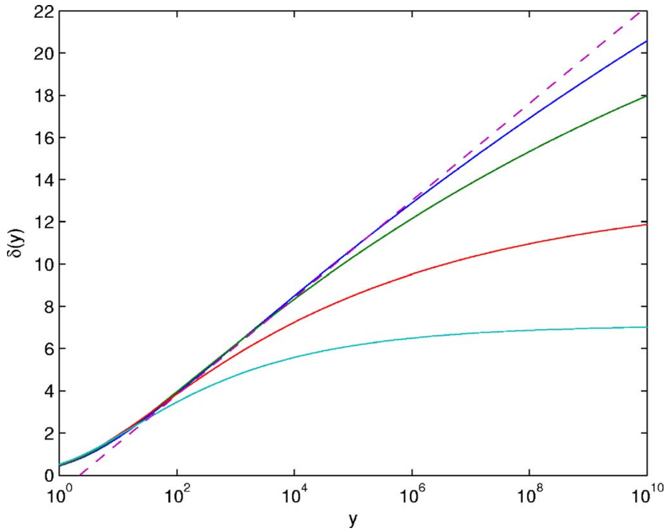

FIG. 8. (Color online) Dependence of $\delta(y)$ given by Eq. (49) as a function of y in logarithmic scale for $\varphi=0.01, 0.02, 0.05, 0.1$ (top to bottom). The straight dashed line corresponds to the logarithmic dependence $\ln y - 0.8$. Not surprisingly, the closer φ is to 0, the larger is the range of $\ln y$ over which the approximation $\delta(y) \sim \ln y$ holds.

dimensionless variable y, but only in some intermediate scaling range. Nonconcavity also prevents obtaining the exact multifractal spectrum of dimensions $f(\alpha)$ of singularities α, but yields only a concave envelope of it [38].

VI. NUMERICAL AND MATHEMATICAL INSIGHTS ON THE ORIGIN OF THE EFFECTIVE MULTIFRACTALITY

As we showed, these multifractal properties are only effective properties or approximations, as exact multifractality only holds when $C(\tau)$ defined in Eq. (14) is proportional to the logarithm of τ (up to an integral scale). The approximate multifractality discussed here can be tracked back to the approximate logarithmic dependence of $C(\tau)$, as we now make clear, numerically and mathematically.

Let us consider the second-order moment $S_2(y)$. The scaling (17) is a precise description of $S_2(y)$ if $G(y)$ in Eq. (13) is close to a power law, i.e., $\sigma^2 d(y)$ is close to $\lambda^2 \ln y$. To test if this is the case for different values of σ^2 and φ, we construct

$$\delta(y) = \sigma^2 d(y)/\lambda^2, \tag{48}$$

which should be close to $\ln y$ to justify our results above. Recall that, for rather small σ^2 (actually, we just need that $\sigma^2 \lesssim 100$), we have the relation (44) with Eq. (45), which together with Eq. (48) yields

$$\delta(y) = \frac{d(y)}{b(\varphi)} = \frac{1 - C(y)}{b(\varphi)}, \tag{49}$$

which has to be almost equal to $\ln y$ for any σ^2 and φ to justify our results. This is verified in Fig. 8, which plots $\delta(y)$ given by Eq. (49). As φ departs more and more from 0, increasing deviations from $\ln y$ occur, which reduce the range over which scaling and multifractality can be observed. An argument similar to that presented in Fig. 8 was proposed in [18,19] for the multiscaling of the conditional response function (Omori law).

This argument can be made mathematically precise as follows. Let us consider the function

$$K(y,\varphi) = \int_0^\infty h(x,\varphi) h(x+y,\varphi) dx, \tag{50}$$

where we redefine

$$h(x,\varphi) = \frac{1}{(1+x)^{\varphi+1/2}}, \tag{51}$$

such that

$$\sigma^2 \equiv K(0,\varphi) = \frac{1}{2\varphi}. \tag{52}$$

Determining how well the covariance function $C(t)$, defined by Eq. (14), is approximated by a logarithm of t amounts to study how well $K(y,\varphi)$ is approximated by a logarithmic function of y.

The study of $K(y,\varphi)$ is based on its explicit analytical expression

$$K(y,\varphi) = \frac{\sqrt{\pi}\,\Gamma(\varphi)\sec(\pi\varphi)}{2\Gamma(\varphi+1/2)}\left(-\frac{2}{y}\right)^{2\varphi} - 2\frac{(1+y)^{-\varphi+1/2}}{1-2\varphi} \times F\left(1, \frac{1}{2}+\varphi, \frac{3}{2}-\varphi, 1+y\right), \tag{53}$$

where $F(a,b,c,z)$ is a hypergeometric function.

Let us introduce the analog of the function $d(y)$ [see Eq. (13)] defined by

$$D(y,\varphi) = \sigma^2 - K(y,\varphi) = \frac{1}{2\varphi} - K(y,\varphi). \tag{54}$$

Using Eq. (53), we obtain the following asymptotic expression for $D(y,\varphi)$:

$$D(y,\varphi) \simeq -\frac{1}{2}\frac{F(y,\varphi)-F(y,0)}{\varphi} + \ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right), \tag{55}$$

valid for $\varphi \to 0$, where

$$F(y,\varphi) = \rho(\varphi)\left(\frac{2}{y}\right)^{2\varphi}, \tag{56}$$

and

$$\rho(\varphi) = \varphi\Gamma(\varphi)\frac{\sqrt{\pi}}{\Gamma(1/2+\varphi)}\sec(\pi\varphi). \tag{57}$$

We verify that the asymptotic expression (55) gives an excellent approximation to the exact expression for $D(y,\varphi)$ obtained by using Eq. (53) for $1 \leqslant y \leqslant 10^{15}$ for $\varphi \lesssim 0.1$.

To reveal the dependence of $D(y,\varphi)$ with respect to y for small φ, we may interpret the first fraction in the right-hand side of the relation (55) as a discrete approximation of the derivative $F'(y,\varphi)$ of the function $F(y,\varphi)$ with respect to φ. This leads to the following approximate relation:

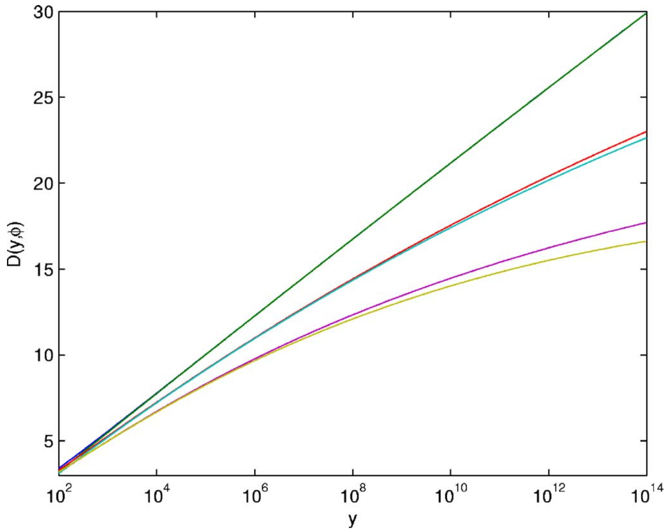

FIG. 9. (Color online) Plots of the exact function $D(y,\varphi)$ and its approximation (61) for $\varphi=0.001,0.01,0.02$ (top to bottom).

$$D(y,\varphi) \simeq -\frac{1}{2}F'\left(y,\frac{\varphi}{2}\right) + \ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right), \tag{58}$$

where

$$F'(y,\varphi) = -2\ln[A(\varphi)y^{\rho(\varphi)}]\left(\frac{2}{y}\right)^{2\varphi}, \tag{59}$$

with

$$A(\varphi) = \left(\frac{1}{2}\right)^{\rho(\varphi)} \exp\left(-\frac{\rho'(\varphi)}{2}\right). \tag{60}$$

Since $\ln[(\sqrt{1+y}+1)/(\sqrt{1+y}-1)]=2/\sqrt{y}+O(1/y)$ for $y \gg 1$, we can neglect this logarithmic term in Eq. (58) for $\varphi < 1/2$ and obtain the following logarithmic-power approximation

$$D(y,\varphi) \simeq \ln[A(\varphi/2)y^{\rho(\varphi/2)}]\left(\frac{2}{y}\right)^{\varphi}. \tag{61}$$

Figure 9 plots the exact function $D(y,\varphi)$ and its approximation (61), showing the approximate linear dependence of $D(y,\varphi)$ as a function of $\ln y$ for a large range of y for $\varphi > 0$.

For $\varphi \to 0$, $\rho(\varphi/2) \to 1$ and we recover $D(y,\varphi \to 0) \to \ln y$, as expected. For $\varphi > 0$, expression (61) makes explicit the deviation from a pure logarithmic dependence of $D(y,\varphi)$ in the form of the power factor $(2/y)^{\varphi}$. The predicted deviation from a pure logarithm gives us the possibility to estimate the integral scale beyond which the effective multifractality breaks down. We thus define the integral scale L_{ϵ} as the solution of the equation

$$\left(\frac{2}{y}\right)^{\varphi} = \epsilon, \tag{62}$$

where $\epsilon < 1$ measures the deviation of $D(y,\varphi)$ from $\ln y$. This gives $L_{\epsilon}=2/\epsilon^{1/\varphi}$. Taking arbitrarily $\epsilon=1/2$, we obtain $L_{1/2} \sim 2000$ for $\varphi=0.1$, $L_{1/2} \sim 1.3\times 10^{30}$ for $\varphi=0.01$, and the astronomical value $L_{1/2} \sim 10^{301}$ for $\varphi=0.001$. These values explain why the saturation of the effective multifractal scaling predicted for large y is not observed for small φ's.

VII. CONCLUDING REMARKS

We have confirmed on the multifractal spectrum $\zeta(q)$ the proposition previously introduced in the context of a model of earthquake [18,19] that processes constructed as exponentials of long-memory processes should exhibit multifractal properties over a significant range of the parameters. In this sense, while capturing some of the main observed properties of the stochastic financial volatility, the exponential Ornstein-Uhlenbeck model [39] falls short of explaining its multifractal properties, due to the existence of only one time scale in the Ornstein-Uhlenbeck process inside the exponential, in contrast with the power law structure of the dependence of the ω process in the MRW and in our model. The initial argument of Ouillon and Sornette concerned the time decay of the conditional expectation of the response function, called the Omori law for the decay of the number of aftershocks in the context of seismology. This conditional temporal multifractal response function obtained in [18,19] generalized the prediction for the multifractal random walk discussed in Ref. [15], which was tested empirically on financial time series. Here, we have extended these analyses to show that multifractality quantified in terms of structure functions is a robust property of this class of processes defined as exponentials of long memory processes.

Notwithstanding the fact that the multifractal properties discussed here are only effective properties or approximations, we claim that there is probably no way to tell the difference between an exact multifractal random walk or an exact multifractal scaling from our approximate scaling laws and approximate $\zeta(q)$ functions, given the noise and the limited ranges usually seen in experimental and numerical studies [34–36]. We have illustrated this claim by showing that a given multifractal spectrum $\zeta(q)$ can be obtained for several sets of parameters of our model.

What are then the differences between the MRW and our model (5) with Eqs. (6) and (7)?

First, as we made clear before, one might be able to distinguish the MRW, which has exact scaling, from the model (5), if inhuman precision and a very large range of scales are available.

Second, the MRW is exactly multifractal up to the integral time scale T, which is absent (or pushed to infinity) in our model. The detection of such an integral time scale thus provides *a priori* an important test for falsifying either the MRW or our model. However, in another work with J.-F. Muzy (in preparation), we will show that this is more easily said than done, as any attempt to measure the integral scale T actually fails to provide the correct value and only gives an apparent value which is comparable to the length L of the time series, as long as T is larger than L. As we will show, this result casts strong doubts on the validity of the determinations of the integral scales in previous calibrations of the MRW to empirical data.

Third, as we said above, the MRW has all its moments of order $q > q^{\dagger}$, where $q^{\dagger} \sim 1/\lambda$ is the solution of $D[\alpha(q^{\dagger})]=0$

[31], which is infinite. This signals that the probability density function (PDF) of the increments of the MRW process is heavy-tailed with an exponent smaller than (or equal to) q^*. In contrast, our model has all its moment finite. As discussed above, it is not yet clear whether the two models give PDFs which can be differentiated, given the limited quality and size of available data.

Sornette, Malevergne, and Muzy have shown that the conditional relaxation of the MRW increments after a local peak is a power law with an exponent $\alpha(s)$ being a continuous function of the log-amplitude s of the peak [15]. Ouillon and Sornette have shown that the same property holds for models generalizing Eq. (5) with Eqs. (6) and (7) by having stochastic increments dW distributed according to a power law with tail exponent μ, as long as the relation $(\varphi+1/2)\mu=1$ holds [18,19], where the MRW case is recovered for $\varphi=0$ and $\mu=2$. In another work with J.-F. Muzy, we will show that the conditional relaxation departs from a power law when $\varphi>0$ but, as for the other properties, this departure is weak and difficult to distinguish from the pure power law result obtained for the MRW.

In summary, while the exact mathematical properties of the MRW and of our model (5) with Eqs. (6) and (7) are no doubt different, our model approaches arbitrarily closely the MRW for $\varphi\to 0$ and empirically distinguishing the two models may be beyond reach even for $\varphi\propto 0.1$ or larger, given the limited size and noisy nature of available time series.

ACKNOWLEDGMENTS

We acknowledge stimulating discussions with J.-F. Muzy and G. Ouillon and thank J.-F. Muzy for a critical reading of the manuscript. This work was partially supported by NSF-EAR02-30429, and by the Southern California Earthquake Center (SCEC). SCEC is funded by NSF Cooperative Agreement EAR-0106924 and USGS Cooperative Agreement 02HQAG0008. The SCEC contribution number for this paper is 996.

[1] L. F. Richardson, *Weather Prediction by Numerical Process* (Cambridge University Press, Cambridge, 1922).
[2] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **31**, 99 (1941) (in Russian).
[3] B. B. Mandelbrot, *On Intermittent Free Turbulence, Turbulence of Fluids and Plasmas*, edited by Ernst Weber (Interscience, New York 1969).
[4] B. B. Mandelbrot, C.R. Seances Acad. Sci., Ser. A **278A**, 289 (1974); **278A**, 355 (1974).
[5] U. Frisch, *Turbulence, The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[6] B. B. Mandelbrot and R. Hudson, *The (Mis)Behaviour of Markets* (Perseus Books, Cambridge, MA 2004).
[7] A. Arneodo, E. Bacry, and J. F. Muzy, J. Math. Phys. **39**, 4142 (1998).
[8] D. Sornette, Phys. Rep. **297**, 239 (1998).
[9] A. Arneodo, J. F. Muzy, and D. Sornette, Eur. Phys. J. B **2**, 277 (1998).
[10] D. Sornette and W.-X. Zhou, e-print cond-mat/0503607.
[11] E. Bacry, J. Delour, and J.-F. Muzy, Phys. Rev. E **64**, 026103 (2001).
[12] F. G. Schmitt and D. Marsan, Eur. Phys. J. B **20**, 3 (2001).
[13] J.-F. Muzy and E. Bacry, Phys. Rev. E **66**, 056121 (2002).
[14] E. Bacry and J.-F. Muzy, Commun. Math. Phys. **236**, 449 (2003).
[15] D. Sornette, Y. Malevergne, and J.-F. Muzy, J. Risk **16**, 67 (2003).
[16] F. G. Schmitt, Eur. Phys. J. B **34**, 85 (2003).
[17] J. Barral, and B. B. Mandelbrot, *Proceedings of Symposia on Pure Mathematics: 72, Part 2: Multifractals, Probability and Statistical Mechanics, Applications*, edited by Michel L. Lapidus and Machiel van Frankenhuijsen (American Mathematical Society, Providence, RI 2004) pp. 3–16; (2005), pp. 17–52.
[18] D. Sornette and G. Ouillon, Phys. Rev. Lett. **94**, 038501 (2005).
[19] G. Ouillon and D. Sornette, J. Geophys. Res. **110**, B04306 (2005).
[20] J. Beran, *Monographs on Statistics and Applied Probability 61* (Chapman and Hall, New York, 1994).
[21] R. F. Baillie, T. Bollerslev, and H. O. Mikkelsen, J. Econometr. **74**, 3 (1996).
[22] R. Baillie, J. Econometr. **73**, 5 (1996).
[23] B. B. Mandelbrot, *Fractals and Scaling in Finance* (Springer, New York, 1997); A. Fisher, L. Calvet, and B. B. Mandelbrot, *Multifractality of DEM/$ rates*, Cowles Foundation Discussion Paper 1165 (1997).
[24] B. B. Mandelbrot, Sci. Am. **February**, 70 (1999).
[25] F. Schmitt, D. Schertzer, S. Lovejoy, Appl. Stochas. Mod. Data Anal. **15**, 29 (1999).
[26] M.-E. Brachet, E. Taflin, and J. M. Tcheou, e-print cond-mat/9905169.
[27] T. Lux, Quant. Finance **1**, 632 (2001); in revision for the J. Bus. Econ. Stat. (to be published).
[28] L. Calvet and A. Fisher, Rev. Econ. Stat. **84**, 381 (2002).
[29] L. Calvet and A. Fisher, J. Econometr. **105**, 27 (2001); J. Financ. Econ. **2**, 49 (2004).
[30] J.-P. Bouchaud, M. Potters, and M. Meyer, Eur. Phys. J. B **13**, 595 (2000).
[31] J.-F. Muzy, E. Bacry, and A. Kozhemyak, e-print cond-mat/0509357.
[32] Y. Malevergne, V. F. Pisarenko, and D. Sornette, Quant. Finance **5**, 379 (2005).
[33] Y. Malevergne and D. Sornette, *Extreme Financial Risks (From Dependence to Risk Management)* (Springer, Heidelberg, 2005).
[34] B. B. Mandelbrot, Science **279**, 783 (1998).
[35] O. Biham, O. Malcai, D. A. Lidar, and D. Avnir, Science **279**, 785 (1998).
[36] O. Biham, O. Malcai, D. A. Lidar, and D. Avnir, Science **279**, 1615 (1998).
[37] For the sake of conciseness, we do not show the corresponding figures which are available upon request from the authors.
[38] H. Touchette and C. Beck, e-print cond-mat/0507379.
[39] J. Perell and J. Masoliver, e-print cond-mat/0501639, Quant. Finance (to be published).